Any entire linear combination of these functions will accordingly also be a solution. Such a combination is

$$\frac{(-1)^n J_{-n+\Delta n}(x) - J_{n-\Delta n}(x)}{2\Delta n}.$$

Adding to the numerator of this fraction the null expression $J_n(x) - (-1)^n J_{-n}(x)$ will not affect it, so that we may write our solution

$$\frac{(-1)^{n+1}[J_{-u}(x) - J_{-n+\Delta n}(x)] + J_n(x) - J_{n-\Delta n}(x)}{2\Delta n}.$$

If, now, we let Δn approach zero, we obtain

$$K_n(x) = \frac{1}{2}\left(\frac{dJ_n(x)}{dn} + (-1)^{n+1}\frac{dJ_{-n}(x)}{dn}\right)$$

as a second solution of Bessel's equation *when n is a positive integer.* This function is called a Bessel's function of the second kind of order n.

Now we obtain at once by differentiation

$$\frac{dJ_n(x)}{dn} = \log\left(\frac{x}{2}\right)J_n(x) + \left(\frac{x}{2}\right)^n \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p} \frac{d[\Gamma(n+p-1)]^{-1}}{dn},$$

$$\frac{dJ_{-n}(x)}{dn} = -\log\left(\frac{x}{2}\right)J_{-n}(x) + \left(\frac{x}{2}\right)^{-n} \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p} \frac{d[\Gamma(-n+p+1)]^{-1}}{dn}.$$

In order to differentiate the reciprocals of $\Gamma(n+p+1)$ and $\Gamma(-n+p+1)$ with respect to n we will introduce a new function using a notation similar to that of Gauss, and write

$$\psi(x) = \frac{d\log\Gamma(x)}{dx}.$$

This function ψ will evidently have the property

$$\psi(x) + \frac{1}{x} = \psi(x+1);$$

so that if x is a positive integer,

$$\psi(x) = \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{3} + \frac{1}{2} + 1 + \psi(1),$$

and $\psi(1)$ has approximately the value -0.5772. In terms of this function we shall have

$$\frac{d\Gamma(x)}{dx} = \Gamma(x)\cdot\psi(x),$$